Finite-time and infinite-time ruin probabilities in a two-dimensional delayed renewal risk model with Sarmanov dependent claims

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Abstract

Consider a two-dimensional delayed renewal risk model with a constant interest rate, where the claim sizes of the two classes form a sequence of independent and identically distributed random vectors following a common bivariate Sarmanov distribution. In the presence of heavy-tailed claim sizes, some asymptotic formulas are derived for the finite-time and infinite-time ruin probabilities.

Keywords: Asymptotics; two-dimensional delayed renewal risk model; finite-time and infinite-time ruin probabilities; bivariate Sarmanov dependence; subexponentiality; extended regular variation

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1 Introduction

In this paper, we consider a two-dimensional delayed renewal risk model with a constant interest rate and dependent claims, where an insurance company operates two lines of business. Each line is assumed to be exposed to catastrophic risks like earthquakes, floods or terrorist attacks. Such risks may affect the two lines of the company at the same time, so the two lines of business share a common claim-number process and some dependence structure may exist between them. In such a model, the claim sizes of the two classes $\{\mathbf{X}_i = (X_{1i}, X_{2i})^{\mathrm{T}},\ i \geq 1\}$ form a sequence of independent, identically distributed (i.i.d.) and nonnegative random vectors with a generic random vector $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$, whose components are however dependent and have marginal distribution functions (d.f's) F_1 and F_2, respectively; and the claim inter-arrival times $\{\theta_i,\ i \geq 1\}$, independent of $\{\mathbf{X}_i,\ i \geq 1\}$, are another sequence of independent, nonnegative and nondegenerate random variables (r.v's). If $\{\theta_i,\ i \geq 2\}$ are identically distributed with common d.f. G, and θ_1 has an arbitrary d.f. G_1, which need not be equal to G (one may have (partial) information on the process before time 0), then the successive claim arrival times, denoted by

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$\{\tau_n = \sum_{i=1}^{n} \theta_i,\ n \geq 1\}$, constitute a delayed renewal counting process

$$N(t) = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \leq t\}},\ t \geq 0, \tag{1.1}$$

with a finite mean function $\lambda(t) = \mathbb{E}N(t) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_i \leq t)$, where $\mathbb{I}_A$ denotes the indicator function of an event A. If $G_1 = G$, then the model is reduced to a two-dimensional (zero-delayed) renewal risk model. The vector of the total premium accumulated up to time $t \geq 0$, denoted by $\mathbf{C}(t) = (C_1(t), C_2(t))^{\mathrm{T}}$ with $\mathbf{C}(0) = (0,0)^{\mathrm{T}}$ and $C_i(t) < \infty,\ i = 1,2$, almost surely for every $t > 0$, is a nonnegative and nondecreasing two-dimensional stochastic process. Assume that $\{\mathbf{X}_i,\ i \geq 1\}$, $\{\theta_i,\ i \geq 1\}$ and $\{\mathbf{C}(t),\ t \geq 0\}$ are mutually independent. Let $\delta \geq 0$ be the constant interest rate, that is to say, at time t one dollar accumulates to $e^{\delta t}$ dollars; and $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ denotes the initial surplus vector. In this two-dimensional setting, the discounted surplus process up to time $t \geq 0$, denoted by $\mathbf{D}(t) = (D_1(t), D_2(t))^{\mathrm{T}}$ has the form

$$\mathbf{D}(t) = \mathbf{x} + \int_{0-}^{t} e^{-\delta s} \mathbf{C}(\mathrm{d}s) - \sum_{i=1}^{N(t)} \mathbf{X}_i e^{-\delta \tau_i}. \tag{1.2}$$

In the one-dimensional setting, the finite-time and infinite-time ruin probabilities are defined, respectively, as, for some finite $T > 0$,

$$\psi(x_1; T) = \mathbb{P}(D_1(t) < 0 \text{ for some } t \in [0, T] | D_1(0) = x_1)$$

and

$$\psi(x_1; \infty) = \mathbb{P}(D_1(t) < 0 \text{ for some } t \in [0, \infty) | D_1(0) = x_1).$$

However, in the two-dimensional case, there are several definitions of ruin; for example, see Cai and Li (2005, 2007).

In this paper, we investigate the ruin probabilities by means of the ruin time

$$T_{\max} = \inf\{t > 0 : \max\{D_1(t), D_2(t)\} < 0\},$$

by convention, $\inf \emptyset = \infty$, which was also investigated by Li et al. (2007). Then, the finite-time ruin probability within a finite time $t > 0$ and the infinite-time ruin probability can be defined, respectively, as

$$\psi(\mathbf{x}; t) = \mathbb{P}(T_{\max} \leq t | \mathbf{D}(0) = \mathbf{x}) \tag{1.3}$$

and

$$\psi(\mathbf{x}; \infty) = \mathbb{P}(T_{\max} < \infty | \mathbf{D}(0) = \mathbf{x}). \tag{1.4}$$

In the one-dimensional case, many works have been devoted to studying the asymptotic behavior of finite-time and infinite-time ruin probabilities. In the one-dimensional renewal risk model with $\delta > 0$ and constant premium rate, Klüppelberg and Stadtmüller (1998) obtained the following result: as $x_1 \to \infty$,

$$\psi(x_1; t) \sim \int_{0-}^{t} \overline{F_1}(x_1 e^{\delta u}) \lambda(\mathrm{d}u), \tag{1.5}$$

with $t = \infty$ for the special case when $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process and F_1 is regularly varying (see the definition below), where $\overline{F_1}(x) = \mathbb{P}(X_1 > x)$. In this special case, Tang (2005) obtained the uniformity on the set $(0, \infty]$ of (1.5), that is,

$$\lim_{x_1 \to \infty} \sup_{t \in (0, \infty]} \left| \frac{\psi(x_1; t)}{\int_{0-}^{t} \overline{F_1}(x_1 e^{\delta u}) \lambda(\mathrm{d}u)} - 1 \right| = 0.$$

Tang (2007) and Chen and Ng (2007) both considered a renewal risk model with extended regularly varying claims. The former obtained the asymptotic relation (1.5) which holds uniformly for all $t \in \Lambda = \{t : \lambda(t) > 0\}$; while the latter only dealt with the case of $t = \infty$, but some dependence structure among claims was considered. Recently, Wang (2008) investigated the asymptotics for finite-time ruin probability in a delayed renewal risk model, when the claim size distribution is subexponential. Wang et al. (2013) studied a dependent renewal risk model with $\delta \geq 0$ and long tailed and dominated varying tailed claims, and showed that (1.5) holds uniformly for all $t \in \Lambda \cap [0, T]$ and any fixed $T \in \Lambda$. Note that the last two results also hold for a stochastic premium process $C_1(t)$.

In the past decade, the investigation of multi-dimensional risk models has attracted a vast amount of attention due to their practical importance. Ruin for multi-dimensional heavy-tailed processes was initially studied by Hult et al. (2005), who mainly focused on multivariate regularly varying random walks and provided sharp asymptotics for general ruin boundaries. Yuen et al. (2006) considered a two-dimensional compound Poisson risk model with $\delta = 0$ and a constant premium rate, and discussed various methods for evaluation of finite-time ruin probability. Li et al. (2007) extended the model by adding Brownian perturbation, and derived the asymptotics for finite-time ruin probability when F_1 and F_2 are both subexponential. Moreover, Chen et al. (2011) established asymptotic formulas for two types of finite-time ruin probabilities for a two-dimensional renewal risk model with heavy-tailed claims; and Chen et al. (2013) studied two kinds of nonstandard two-dimensional models in which the claims among the same line of business are dependent.

In this paper, we aim at studying another kind of two-dimensional risk model, where some dependence may exist between the claims of the two lines. Precisely speaking, we assume that $\{(X_{1i}, X_{2i})^{\mathrm{T}},\ i \geq 1\}$ are i.i.d. random vectors with a generic random pair $(X_1, X_2)^{\mathrm{T}}$ whose components are dependent. A typical example in motor insurance is that a car accident could cause claims for both the vehicle damage and the injuries of the driver and passengers. This example shows some practical relevance of the two-dimensional risk model of study in insurance. The motivation of this study also comes from a recent work by Yang and Li (2014), who modelled the dependence structure of $(X_1, X_2)^{\mathrm{T}}$ by a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution given as

$$\Pi(x_1, x_2) = F_1(x_1)F_2(x_2)(1 + r\overline{F_1}(x_1)\overline{F_2}(x_2)), \quad r \in [-1, 1]. \tag{1.6}$$

In addition, it is worth to note that most of the existing results for two-dimensional risk models are related to finite-time ruin probability. Perhaps, due to its complexity, only a few carry out investigation of infinite-time ruin probability.

In the present paper, we shall use a more general bivariate Sarmanov dependence structure to model $(X_1, X_2)^{\mathrm{T}}$, and investigate the asymptotic behavior of both finite-time and infinite-time ruin probabilities under the conditions that the claim size distributions are subexponential and extended varying tailed, respectively. Additionally, a delayed renewal claim-number process and general premium processes are also considered. The asymptotic formulas derived here successfully capture the impact of the underlying dependence structure of $(X_1, X_2)^{\mathrm{T}}$.

The rest of this paper consists of three sections. Section 2 prepares preliminaries of heavy-tailed distributions and the Sarmanov dependence structure, and presents the two main results. Sections 3 and 4 first state a few lemmas, and then give the proofs of the two results.

2 Preliminaries and main results

Throughout this paper, all limit relationships hold for $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ tending to $(\infty, \infty)^{\mathrm{T}}$ unless stated otherwise. For two positive bivariate functions $f(x_1, x_2)$ and $g(x_1, x_2)$, we write $f \lesssim g$ or

$g \gtrsim f$ if $\limsup f/g \le 1$; write $f \sim g$ if both $f \lesssim g$ and $f \gtrsim g$; write $f(x_1, x_2) = o(g(x_1, x_2))$ if $\lim f(x_1, x_2)/g(x_1, x_2) = 0$; and write $f(x_1, x_2) = O(g(x_1, x_2))$ if $\limsup f(x_1, x_2)/g(x_1, x_2) < \infty$. To avoid triviality, a nonnegative r.v. is always assumed to be nondegenerate at 0. Hereafter, K represents a positive constant whose value may vary from line to line.

We shall restrict claim size distributions to be heavy-tailed. A d.f. V on $[0, \infty)$ is said to be subexponential, denoted by $V \in \mathscr{S}$, if $\overline{V}(x) > 0$ for all $x \ge 0$ and

$$\lim_{x\to\infty} \frac{\overline{V^{2*}}(x)}{\overline{V}(x)} = 2,$$

where V^{2*} denotes the two-fold convolution of V. It is well known that every subexponential d.f. V is long tailed, denoted by $V \in \mathscr{L}$, in the sense that the relation $\overline{V}(x+y) \sim \overline{V}(x),\ x \to \infty$, holds for any $y \in \mathbb{R}$. One of the most important subclass of $\mathscr{S}$ is the class of d.f's with extended regularly varying tails. By definition, a d.f. V on $\mathbb{R}$ is said to be extended regularly varying tailed, if there are some constants $0 < \alpha \le \beta < \infty$, such that for all $0 < y \le 1$,

$$y^{-\alpha} \le \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \le \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \le y^{-\beta},$$

denoted by $V \in \mathrm{ERV}(-\alpha, -\beta)$. In particular, if $\alpha = \beta$, then the class $\mathrm{ERV}(-\alpha, -\beta)$ reduces to the famous class $\mathscr{R}_{-\alpha}$ of d.f's with regularly varying tails. Another useful class consists of all d.f's with dominated variation. A d.f. V on $\mathbb{R}$ is said to be dominatedly varying tailed, denoted by $V \in \mathscr{D}$, if $\overline{V}(xy) = O(\overline{V}(x)),\ x \to \infty$, for any $0 < y < 1$. From Proposition 2.2.1 of Bingham et al. (1987) or Section 3.3 of Tang and Tsitsiashvili (2003a), the following proposition is valid.

Proposition 2.1. *Let a d.f.* $V \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \le \beta < \infty$.

(1) For any $0 < \alpha' < \alpha \le \beta < \beta' < \infty$, *there are two positive constants* c_V *and* d_V, *such that the inequalities*

$$c_V^{-1}\left(\frac{y}{x}\right)^{-\alpha'} \le \frac{\overline{V}(y)}{\overline{V}(x)} \le c_V\left(\frac{y}{x}\right)^{-\beta'}$$

hold for all $x \ge y \ge d_V$.

(2) For any $\beta' > \beta$, *it holds that* $x^{-\beta'} = o(\overline{V}(x))$ *as* $x \to \infty$.

We next introduce a bivariate Sarmanov distribution to model the dependence structure of $(X_1, X_2)^{\mathrm{T}}$. Recall that a bivariate Sarmanov distribution is of the form

$$\mathbb{P}(X_1 \in \mathrm{d}x_1, X_2 \in \mathrm{d}x_2) = (1 + r\phi_1(x_1)\phi_2(x_2))F_1(\mathrm{d}x_1)F_2(\mathrm{d}x_2),\ x_1 \ge 0,\ x_2 \ge 0, \tag{2.1}$$

where the kernels ϕ_1 and ϕ_2 are two functions and the parameter r is a real constant satisfying

$$\mathbb{E}\phi_1(X_1) = \mathbb{E}\phi_2(X_2) = 0, \tag{2.2}$$

and

$$1 + r\phi_1(x_1)\phi_2(x_2) \ge 0 \ \text{ for all } x_1 \in D_{X_1},\ x_2 \in D_{X_2},$$

where $D_{X_1} = \{x_1 \ge 0 : \ \mathbb{P}(X_1 \in (x_1 - \Delta, x_1 + \Delta)) > 0 \text{ for all } \Delta > 0\}$ and $D_{X_2} = \{x_2 \ge 0 : \mathbb{P}(X_2 \in (x_2 - \Delta, x_2 + \Delta)) > 0 \text{ for all } \Delta > 0\}$. Clearly, if $r = 0$ or $\phi_1(x_1) \equiv 0,\ x_1 \in D_{X_1}$, or $\phi_2(x_2) \equiv 0,\ X_2 \in D_{X_2}$, then X_1 and X_2 are independent. In the independent case, without loss of generality, assume that $r = 0$ and $\phi_i(x_i) \equiv 0,\ x_i \in D_{X_i},\ i = 1, 2$. Otherwise, we say that a random vector $(X_1, X_2)^{\mathrm{T}}$ follows a proper bivariate Sarmanov distribution. For some more details on multivariate Sarmanov distributions, one can be referred to Lee (1996) and Kotz et al. (2000) among others. As in Yang and Wang (2013), three common choices for the kernels ϕ_1 and ϕ_2 are listed below:

(a1) $\phi_1(x_1) = 1 - 2F_1(x_1)$ and $\phi_2(x_2) = 1 - 2F_2(x_2)$ for all $x_1 \in D_{X_1}$ and $x_2 \in D_{X_2}$, leading to the FGM distribution given by (1.6).

(a2) $\phi_1(x_1) = e^{-x_1} - \mathbb{E}e^{-X_1}$ and $\phi_2(x_2) = e^{-x_2} - \mathbb{E}e^{-x_2}$ for all $x_1 \in D_{X_1}$ and $x_2 \in D_{X_2}$;

(a3) $\phi_1(x_1) = x_1^p - \mathbb{E}X_1^p$ and $\phi_2(x_2) = x_2^p - \mathbb{E}X_2^p$ for some $p > 0$ and all $x_1 \in D_{X_1}$ and $x_2 \in D_{X_2}$.

The following proposition shows that the kernels are bounded for any proper bivariate Sarmanov distribution, which is due to Yang and Wang (2013).

Proposition 2.2. *Assume that $(X_1, X_2)^{\mathrm{T}}$ follows a proper bivariate Sarmanov distribution of the form (2.1). Then there exist two positive constants b_1 and b_2 such that $|\phi_1(x_1)| \le b_1$ for all $x_1 \in D_{X_1}$ and $|\phi_2(x_2)| \le b_2$ for all $x_2 \in D_{X_2}$.*

Before presenting our two main results, we need to introduce a zero-delayed renewal counting process, corresponding to $N(t)$ defined in (1.1). For any $t \ge 0$, denote

$$N_0(t) = \sum_{n=2}^{\infty} \mathbb{I}_{\{\tau_n - \tau_1 \le t\}},$$

with a finite mean function $\lambda_0(t) = \mathbb{E}N_0(t)$.

Now we are ready to state the main results of this paper, which provide two asymptotic formulas for finite-time and infinite-time ruin probabilities, respectively.

Theorem 2.1. *Consider a two-dimensional delayed renewal risk model with a constant interest rate $\delta \ge 0$ described in Section 1. Let $(X_1, X_2)^{\mathrm{T}}$ follow a bivariate Sarmanov distribution of the form (2.1) with marginal d.f's $F_1 \in \mathscr{S}$ and $F_2 \in \mathscr{S}$. Assume that the limits $\lim_{x\to\infty} \phi_i(x) = d_i$, $i = 1, 2$, exist. Let $T \in \Lambda$ be a positive constant. If $1 + rd_1d_2 > 0$ and $\mathbb{E}\rho^{N_0(T)} < \infty$ for some $\rho > 1 + |r|\kappa_1\kappa_2$, where $\kappa_i = \sup_{x\ge 0} |\phi_i(x)|$, $i = 1, 2$, then*

$$\begin{aligned}\psi(\mathbf{x}; T) \quad \sim \quad & \int_{0-}^{T}\int_{0-}^{T-u} \left(\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)}) + \overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})\right)\lambda_0(\mathrm{d}v)\lambda(\mathrm{d}u) \\ & + (1 + rd_1d_2)\int_{0-}^{T} \overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\lambda(\mathrm{d}u). \qquad (2.3)\end{aligned}$$

Remark 2.1. *In the independence case, i.e. $r = 0$, the condition $\mathbb{E}\rho^{N_0(T)} < \infty$ for some $\rho > 1$ is automatically satisfied, since the moment generating function of $N_0(T)$ is analytic in a neighborhood of zero, see Stein (1946).*

Consider a special case that the two claim size distributions are both regularly varying tailed and the claim arrival process is a Poisson process, then a more transparent formula for the finite-time ruin probability can be derived.

Corollary 2.1. *Consider a two-dimensional Poisson risk model with a constant interest rate $\delta \ge 0$ described in Section 1. Let $(X_1, X_2)^{\mathrm{T}}$ follow a bivariate Sarmanov distribution of the form (2.1) with marginal d.f's $F_1 \in \mathscr{R}_{-\alpha_1}$ and $F_2 \in \mathscr{R}_{-\alpha_2}$ for some $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$; and let $\{N(t),\ t \ge 0\}$ be a Poisson process with intensity $\lambda > 0$. Assume that the limits $\lim_{x\to\infty} \phi_i(x) = d_i$, $i = 1, 2$, exist. If $1 + rd_1d_2 > 0$, then for any $T > 0$,*

$$\psi(\mathbf{x}; T) \sim \left(\frac{\lambda^2(1 - e^{-\alpha_1\delta T})(1 - e^{-\alpha_2\delta T})}{\alpha_1\alpha_2\delta^2} + \frac{\lambda(1 + rd_1d_2)(1 - e^{-(\alpha_1+\alpha_2)\delta T})}{(\alpha_1 + \alpha_2)\delta}\right)\overline{F_1}(x_1)\overline{F_2}(x_2),$$

here, by convention, $(1-e^{-\alpha_i\delta T})/(\alpha_i\delta)=T,$ *if* $\alpha_i\delta=0,\ i=1,2;$ *and* $(1-e^{-(\alpha_1+\alpha_2)\delta T})/((\alpha_1+\alpha_2)\delta)=T,$ *if* $(\alpha_1+\alpha_2)\delta=0.$

Proof. Clearly, $\mathbb{E}\rho^{N(T)}<\infty$ for any $\rho>0$, since $\{N(t),\ t\ge 0\}$ is a Poisson process. Hence, relation (2.3) holds. By $F_i\in\mathscr{R}_{-\alpha_i},\ i=1,2,$

$$\overline{F_i}(xy)\sim y^{-\alpha_i}\overline{F_i}(x)$$

holds uniformly for all $y\in[y_1,y_2]$ and any $0<y_1\le y_2<\infty$ as $x\to\infty$. Then, by the dominated convergence theorem, we have

$$\begin{aligned}&\lim\frac{1}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\int_{0-}^{T}\int_{0-}^{T-u}\Big(\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)})+\overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})\Big)\mathrm{d}v\mathrm{d}u\\&=\frac{(1-e^{-\alpha_1\delta T})(1-e^{-\alpha_2\delta T})}{\alpha_1\alpha_2\delta^2},\end{aligned}$$

and

$$\lim\frac{\int_{0-}^{T}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\mathrm{d}u}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=\frac{1-e^{-(\alpha_1+\alpha_2)\delta T}}{(\alpha_1+\alpha_2)\delta},$$

which imply the desired result. □

Theorem 2.2. *Consider a two-dimensional delayed renewal risk model with a constant interest rate* $\delta>0$ *described in Section 1. Let* $(X_1,X_2)^{\mathrm{T}}$ *follow a bivariate Sarmanov distribution of the form (2.1) with marginal d.f's* $F_1\in\mathrm{ERV}(-\alpha_1,-\beta_1)$ *and* $F_2\in\mathrm{ERV}(-\alpha_2,-\beta_2)$ *for some* $0<\alpha_1\le\beta_1<\infty$ *and* $0<\alpha_2\le\beta_2<\infty$. *Assume that the limits* $\lim_{x\to\infty}\phi_i(x)=d_i,\ i=1,2,$ *exist. If* $1+rd_1d_2>0$, *then relation* (2.3) *holds for* $T=\infty$.

Corollary 2.2. *Under the conditions of Theorem 2.2, if, further,* $F_i\in\mathscr{R}_{-\alpha_i},\ \alpha_i>0,\ i=1,2,$ *and* $\{N(t),\ t\ge 0\}$ *is a Poisson process with intensity* $\lambda>0$, *then*

$$\psi(\mathbf{x};\infty)\sim\left(\frac{\lambda^2}{\alpha_1\alpha_2\delta^2}+\frac{\lambda(1+rd_1d_2)}{(\alpha_1+\alpha_2)\delta}\right)\overline{F_1}(x_1)\overline{F_2}(x_2).$$

Proof. By Proposition 2.1 (1), for any $0<\alpha_i'<\alpha_i,\ i=1,2,$ there are some positive constants c_{F_i} and d_{F_i}, such that for all $u\ge 0,\ v\ge 0$ and $x_i\ge d_{F_i},\ i=1,2,$

$$\frac{\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)})}{\overline{F_1}(x_1)\overline{F_2}(x_2)}+\frac{\overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\le c_{F_1}c_{F_2}\Big(e^{-\alpha_1'\delta u-\alpha_2'\delta(u+v)}+e^{-\alpha_1'\delta(u+v)-\alpha_2'\delta u}\Big),$$

which is integrable on $[0,\infty)\times[0,\infty)$. Thus, by the dominated convergence theorem, we have

$$\lim\frac{1}{\overline{F_1}(x_1)\overline{F_2}(x_2)}\int_{0-}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)})+\overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})\Big)\mathrm{d}v\mathrm{d}u=\frac{1}{\alpha_1\alpha_2\delta^2},$$

and

$$\lim\frac{\int_{0-}^{\infty}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\mathrm{d}u}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=\frac{1}{(\alpha_1+\alpha_2)\delta},$$

which lead to the desired result. □

3 Proof of Theorem 2.1

In the sequel, without loss of generality, all proofs of the following theorems and lemmas will be given only in the case that $(X_1, X_2)^{\mathrm{T}}$ follows a proper bivariate Sarmanov distribution, and the proofs in the independence case are trivial. Motivated by Yang and Li (2014), we go along a similar line to complete the proof of Theorem 2.1.

We start this section by a series of lemmas.

Lemma 3.1. *Let $f_1(x_1,x_2) \le f_2(x_1,x_2)$ and $g_1(x_1,x_2) \le g_2(x_1,x_2)$ be four positive bivariate functions on $\mathbb{R}^2$. If $f_i(x_1,x_2) \sim g_i(x_1,x_2),\ i=1,2$, and $g_2(x_1,x_2) = O(g_2(x_1,x_2) - g_1(x_1,x_2))$, then $f_2(x_1,x_2) - f_1(x_1,x_2) \sim g_2(x_1,x_2) - g_1(x_1,x_2)$ holds.*

Proof. For any $\epsilon > 0$, by $f_i(x_1,x_2) \sim g_i(x_1,x_2),\ i=1,2$, there exists some large x_0 such that for all $\min\{x_1,x_2\} \ge x_0$,

$$(1-\epsilon)g_i(x_1,x_2) \le f_i(x_1,x_2) \le (1+\epsilon)g_i(x_1,x_2),$$

$i = 1,2$, which yields

$$1 - \frac{2\epsilon g_2(x_1,x_2)}{g_2(x_1,x_2) - g_1(x_1,x_2)} \le \frac{f_2(x_1,x_2) - f_1(x_1,x_2)}{g_2(x_1,x_2) - g_1(x_1,x_2)} \le 1 + \frac{2\epsilon g_2(x_1,x_2)}{g_2(x_1,x_2) - g_1(x_1,x_2)}. \tag{3.1}$$

Therefore, the desired equivalence follows from (3.1) and by noting $g_2(x_1,x_2) = O(g_2(x_1,x_2) - g_1(x_1,x_2))$, and the arbitrariness of $\epsilon > 0$. □

The second lemma is a combination of Corollary 5.2 and Proposition 5.1 of Tang and Tsitsiashvili (2003b), see also Hao and Tang (2008).

Lemma 3.2. *Let $\{\xi_i,\ 1 \le i \le n\}$ be n independent r.v's with d.f's $\{V_i,\ 1 \le i \le n\}$. If there is a d.f. $V \in \mathscr{S}$ such that $\overline{V_i}(x) \sim l_i \overline{V}(x),\ x \to \infty$, holds for some $l_i > 0,\ 1 \le i \le n$, then for any fixed $0 < a \le b < \infty$, it holds uniformly for all $(c_1,\ldots,c_n) \in [a,b]^n$ that as $x \to \infty$,*

$$\mathbb{P}\Big(\sum_{i=1}^n c_i\xi_i > x\Big) \sim \sum_{i=1}^n \overline{V_i}(x/c_i),$$

that is,

$$\lim_{x\to\infty} \sup_{(c_1,\ldots,c_n)\in[a,b]^n} \left| \frac{\mathbb{P}\Big(\sum_{i=1}^n c_i\xi_i > x\Big)}{\sum_{i=1}^n \overline{V_i}(x/c_i)} - 1 \right| = 0.$$

The following lemma is a bivariate Sarmanov version of Lemma 3.2.

Lemma 3.3. *Let $\{\mathbf{X}_i,\ 1 \le i \le n\}$ be n i.i.d. nonnegative random vectors with a generic random vector $\mathbf{X}$ following a bivariate Sarmanov distribution of the form (2.1) with marginal d.f's $F_1 \in \mathscr{S}$ and $F_2 \in \mathscr{S}$. Assume that the limits $\lim_{x\to\infty}\phi_i(x) = d_i,\ i=1,2$, exist. If $1 + rd_1d_2 > 0$, then for any fixed $0 < a \le b < \infty$, it holds uniformly for all $(c_1,\ldots,c_n) \in [a,b]^n$ that*

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n c_iX_{1i} > x_1, \sum_{j=1}^n c_jX_{2j} > x_2\Big) &\sim \sum_{i=1}^n \sum_{1\le j\ne i\le n} \overline{F_1}\Big(\frac{x_1}{c_i}\Big)\overline{F_2}\Big(\frac{x_2}{c_j}\Big) \\ &\quad +(1+rd_1d_2)\sum_{i=1}^n \overline{F_1}\Big(\frac{x_1}{c_i}\Big)\overline{F_2}\Big(\frac{x_2}{c_i}\Big) \\ &\sim \sum_{i=1}^n\sum_{j=1}^n \mathbb{P}(c_iX_{1i} > x_1, c_jX_{2j} > x_2). \end{aligned} \tag{3.2}$$

Proof. If we have proven the first equivalence in (3.2), then the second one can be easily calculated according to the dependence assumption on $\{\mathbf{X}_i,\ i \geq 1\}$ and $\lim_{x\to\infty} \phi_i(x) = d_i,\ i = 1, 2$. Without loss of generality, we only prove the first relation of (3.2) for $n = 2$.

Let random vector $(X_1^*, X_2^*)^{\mathrm{T}}$ be the independent version of $(X_1, X_2)^{\mathrm{T}}$. By Proposition 2.2, there exist two constants $b_1 > 1$ and $b_2 > 1$ such that $|\phi_i(x_i)| \leq b_i - 1$ for all $x_i \in D_{X_i},\ i = 1, 2$. Obviously, $d_i < b_i,\ i = 1, 2$. Let $\widetilde{X}_1^*$ and $\widetilde{X}_2^*$ be two nonnegative independent r.v's, which are also independent of $(X_1^*, X_2^*)^{\mathrm{T}}$, with d.f's $\widetilde{F}_1$ and $\widetilde{F}_2$, respectively, defined by

$$\widetilde{F}_i(\mathrm{d}x_i) = \left(1 - \frac{\phi_i(x_i)}{b_i}\right) F_i(\mathrm{d}x_i),\ x_i \in D_{X_i},\ i = 1, 2. \tag{3.3}$$

By (2.2) and the conditions of the lemma, clearly, for each $i = 1, 2$, the d.f. $\widetilde{F}_i$ is well defined, and as $x \to \infty$,

$$\overline{\widetilde{F}_i}(x) \sim \left(1 - \frac{d_i}{b_i}\right)\overline{F_i}(x), \tag{3.4}$$

which implies $\widetilde{F}_i \in \mathscr{S}$. Let $(X_{1i}^*, X_{2i}^*)^{\mathrm{T}},\ i = 1, 2$, be the independent copies of $(X_1^*, X_2^*)^{\mathrm{T}}$; let $(\widetilde{X}_{1i}^*, \widetilde{X}_{2i}^*)^{\mathrm{T}},\ i = 1, 2$, be the independent copies of $(\widetilde{X}_1^*, \widetilde{X}_2^*)^{\mathrm{T}}$; and all of the above random vectors and $(X_{1i}, X_{2i})^{\mathrm{T}},\ i = 1, 2$, are mutually independent. Then, we have

$$\begin{aligned}
I &:= \mathbb{P}(c_1X_{11} + c_2X_{12} > x_1, c_1X_{21} + c_2X_{22} > x_2) \\
&= \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}(c_2X_{12} > x_1 - c_1u, c_2X_{22} > x_2 - c_1v)(1 + r\phi_1(u)\phi_2(v))F_1(\mathrm{d}u)F_2(\mathrm{d}v) \\
&= (1 + rb_1b_2)\mathbb{P}(c_1X_{11}^* + c_2X_{12} > x_1, c_1X_{21}^* + c_2X_{22} > x_2) \\
&\quad -rb_1b_2\mathbb{P}(c_1\widetilde{X}_{11}^* + c_2X_{12} > x_1, c_1X_{21}^* + c_2X_{22} > x_2) \\
&\quad -rb_1b_2\mathbb{P}(c_1X_{11}^* + c_2X_{12} > x_1, c_1\widetilde{X}_{21}^* + c_2X_{22} > x_2) \\
&\quad +rb_1b_2\mathbb{P}(c_1\widetilde{X}_{11}^* + c_2X_{12} > x_1, c_1\widetilde{X}_{21}^* + c_2X_{22} > x_2) \\
&=: (1 + rb_1b_2)I_1 - rb_1b_2I_2 - rb_1b_2I_3 + rb_1b_2I_4.
\end{aligned} \tag{3.5}$$

In a same manner, I_1 can be further decomposed into four parts as

$$I_1 = (1 + rb_1b_2)I_{11} - rb_1b_2I_{12} - rb_1b_2I_{13} + rb_1b_2I_{14}, \tag{3.6}$$

where, by Lemma 3.2 and (3.4), it holds that uniformly for all $(c_1, c_2) \in [a, b]^2$,

$$\begin{aligned}
I_{11} &= \mathbb{P}(c_1X_{11}^* + c_2X_{12}^* > x_1)\mathbb{P}(c_1X_{21}^* + c_2X_{22}^* > x_2) \\
&\sim \left(\overline{F_1}\left(\frac{x_1}{c_1}\right) + \overline{F_1}\left(\frac{x_1}{c_2}\right)\right)\left(\overline{F_2}\left(\frac{x_2}{c_1}\right) + \overline{F_2}\left(\frac{x_2}{c_2}\right)\right),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
I_{12} &= \mathbb{P}(c_1X_{11}^* + c_2\widetilde{X}_{12}^* > x_1)\mathbb{P}(c_1X_{21}^* + c_2X_{22}^* > x_2) \\
&\sim \left(\overline{F_1}\left(\frac{x_1}{c_1}\right) + \left(1 - \frac{d_1}{b_1}\right)\overline{F_1}\left(\frac{x_1}{c_2}\right)\right)\left(\overline{F_2}\left(\frac{x_2}{c_1}\right) + \overline{F_2}\left(\frac{x_2}{c_2}\right)\right),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
I_{13} &= \mathbb{P}(c_1X_{11}^* + c_2X_{12}^* > x_1)\mathbb{P}(c_1X_{21}^* + c_2\widetilde{X}_{22}^* > x_2) \\
&\sim \left(\overline{F_1}\left(\frac{x_1}{c_1}\right) + \overline{F_1}\left(\frac{x_1}{c_2}\right)\right)\left(\overline{F_2}\left(\frac{x_2}{c_1}\right) + \left(1 - \frac{d_2}{b_2}\right)\overline{F_2}\left(\frac{x_2}{c_2}\right)\right),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
I_{14} &= \mathbb{P}(c_1X_{11}^* + c_2\widetilde{X}_{12}^* > x_1)\mathbb{P}(c_1X_{21}^* + c_2\widetilde{X}_{22}^* > x_2) \\
&\sim \left(\overline{F_1}\left(\frac{x_1}{c_1}\right) + \left(1 - \frac{d_1}{b_1}\right)\overline{F_1}\left(\frac{x_1}{c_2}\right)\right)\left(\overline{F_2}\left(\frac{x_2}{c_1}\right) + \left(1 - \frac{d_2}{b_2}\right)\overline{F_2}\left(\frac{x_2}{c_2}\right)\right).
\end{aligned} \tag{3.10}$$

In the case $r \geq 0$, by (3.7)–(3.10), we have that uniformly for all $(c_1, c_2) \in [a, b]^2$,

$$\begin{aligned}(1 + rb_1b_2)I_{11} + rb_1b_2I_{14} \sim\ & (1 + 2rb_1b_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + 2rb_1b_2 - rb_1d_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) \\ & +(1 + 2rb_1b_2 - rb_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + 2rb_1b_2 - rb_1d_2 - rb_2d_1 + rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big), \qquad (3.11)\end{aligned}$$

and

$$\begin{aligned}rb_1b_2(I_{12} + I_{13}) \sim\ & 2rb_1b_2\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) + (2rb_1b_2 - rb_1d_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) \\ & +(2rb_1b_2 - rb_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(2rb_1b_2 - rb_1d_2 - rb_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big). \qquad (3.12)\end{aligned}$$

Denote the right-hand sides of (3.11) and (3.12) by $g_2(x_1, x_2)$ and $g_1(x_1, x_2)$ as in Lemma 3.1, respectively. Note that $g_2(x_1, x_2) \geq g_1(x_1, x_2)$,

$$\begin{aligned}g_2(x_1, x_2) - g_1(x_1, x_2) =\ & \overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) + \overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) + \overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) \\ \geq\ & \min\{1, 1 + rd_1d_2\}\sum_{i=1}^{2}\sum_{j=1}^{2}\overline{F_1}\Big(\frac{x_1}{c_i}\Big)\overline{F_2}\Big(\frac{x_2}{c_j}\Big), \qquad (3.13)\end{aligned}$$

and

$$g_2(x_1, x_2) \leq (1 + 5|r|b_1b_2)\sum_{i=1}^{2}\sum_{j=1}^{2}\overline{F_1}\Big(\frac{x_1}{c_i}\Big)\overline{F_2}\Big(\frac{x_2}{c_j}\Big), \qquad (3.14)$$

which imply $g_2(x_1, x_2) = O(g_2(x_1, x_2) - g_1(x_1, x_2))$. Thus, from Lemma 3.1 and (3.6), we obtain that uniformly for all $(c_1, c_2) \in [a, b]^2$,

$$\begin{aligned}I_1 \sim\ & \overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) + \overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) + \overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big). \qquad (3.15)\end{aligned}$$

In the case $r < 0$, by (3.7)–(3.10), we reconsider I_1 as

$$\begin{aligned}I_{11} - rb_1b_2(I_{12} + I_{13}) \sim\ & (1 + 2|r|b_1b_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + 2|r|b_1b_2 - |r|b_1d_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big) \\ & +(1 + 2|r|b_1b_2 - |r|b_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big) \\ & +(1 + 2|r|b_1b_2 - |r|b_1d_2 - |r|b_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big), \qquad (3.16)\end{aligned}$$

and

$$-rb_1b_2(I_{11}+I_{14}) \sim 2|r|b_1b_2\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+(2|r|b_1b_2-|r|b_1d_2)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big)$$
$$+(2|r|b_1b_2-|r|b_2d_1)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)$$
$$+(2|r|b_1b_2-|r|b_1d_2-|r|b_2d_1+|r|d_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big), \quad (3.17)$$

uniformly for all $(c_1,c_2)\in[a,b]^2$. Rewrite the right-hand sides of (3.16) and (3.17) as $g_2(x_1,x_2)$ and $g_1(x_1,x_2)$ as above, respectively. It can be easily seen that relations (3.13) and (3.14) still hold. Then, Lemma 3.1 leads to (3.15).

By the same argument, we can derive that uniformly for all $(c_1,c_2)\in[a,b]^2$,

$$I_2 \sim \Big(1-\frac{d_1}{b_1}\Big)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+\Big(1-\frac{d_1}{b_1}\Big)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big)$$
$$+\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+(1+rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big), \quad (3.18)$$
$$I_3 \sim \Big(1-\frac{d_2}{b_2}\Big)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big)$$
$$+\Big(1-\frac{d_2}{b_2}\Big)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+(1+rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big), \quad (3.19)$$

and

$$I_4 \sim \Big(1-\frac{d_1}{b_1}-\frac{d_2}{b_2}+\frac{d_1d_2}{b_1b_2}\Big)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+\Big(1-\frac{d_1}{b_1}\Big)\overline{F_1}\Big(\frac{x_1}{c_1}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big)$$
$$+\Big(1-\frac{d_2}{b_2}\Big)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_1}\Big)+(1+rd_1d_2)\overline{F_1}\Big(\frac{x_1}{c_2}\Big)\overline{F_2}\Big(\frac{x_2}{c_2}\Big). \quad (3.20)$$

Substituting (3.15) and (3.18)–(3.20) into (3.5), and again using the similar approach to estimate I_1, leads to the first relation of (3.2) for $n=2$. This completes the proof of the lemma. □

The next lemma gives the Kesten's bound for bivariate Sarmanov distributions with both subexponential marginal d.f.'s.

Lemma 3.4. *Under the conditions of Lemma 3.3, for any $\epsilon>0$, there exists a positive constant K_ϵ, such that*

$$\mathbb{P}\Big(\sum_{i=1}^n X_{1i}>x_1,\sum_{j=1}^n X_{2j}>x_2\Big)\le K_\epsilon(1+|r|\kappa_1\kappa_2)^n(1+\epsilon)^n\overline{F_1}(x_1)\overline{F_2}(x_2) \quad (3.21)$$

holds for all $x_1\ge0,\ x_2\ge0$ and $n\ge1$, where $\kappa_i=\sup_{x\ge0}|\phi_i(x)|,\ i=1,2$.

Proof. For all $x_1\ge0,\ x_2\ge0$ and $n\ge1$,

$$\mathbb{P}\Big(\sum_{i=1}^n X_{1i}>x_1,\sum_{j=1}^n X_{2j}>x_2\Big) = \underset{\{\sum_{i=1}^n u_i>x_1,\sum_{i=1}^n v_i>x_2\}}{\int\cdots\int}\prod_{i=1}^n(1+r\phi_1(u_i)\phi_2(v_i))F_1(\mathrm{d}u_i)F_2(\mathrm{d}v_i)$$
$$\le (1+|r|\kappa_1\kappa_2)^n\overline{F_1^{n*}}(x_1)\overline{F_2^{n*}}(x_2),$$

which, by the standard Kesten's inequality (see, e.g., Lemma 1.3.5 (c) of Embrechts et al. (1997)), implies that relation (3.21) holds for all $x_1\ge0,\ x_2\ge0$ and $n\ge1$. □

Lemma 3.5. *Consider a two-dimensional risk model with a constant interest rate $\delta \geq 0$. Let $(X_1, X_2)^{\mathrm{T}}$ follow a bivariate Sarmanov distribution of the form (2.1) with marginal d.f's $F_1 \in \mathscr{S}$ and $F_2 \in \mathscr{S}$; and let $(Z_1, Z_2)^{\mathrm{T}}$ be a nonnegative random vector. Assume that $\{\mathbf{X}_i,\ i \geq 1\}$, $\{N(t),\ t \geq 0\}$ and $(Z_1, Z_2)^{\mathrm{T}}$ are mutually independent, and the limits $\lim_{x\to\infty} \phi_i(x) = d_i,\ i = 1, 2$, exist. Let $T \in \Lambda$ be a positive constant. If $1 + rd_1d_2 > 0$, then for any fixed $n \geq 1$,*

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1 + Z_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2 + Z_2, N(T) = n\Big) \\
&\sim \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2, N(T) = n). \qquad (3.22)
\end{aligned}$$

Proof. By using Lemma 3.3, we derive that

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1 + Z_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2 + Z_2, N(T) = n\Big) \\
&= \int\cdots\int_{\{0\le t_1\le\ldots\le t_n\le T, t_{n+1}>T\}} \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta t_i} > x_1 + z_1, \sum_{j=1}^{n} X_{2j}e^{-\delta t_j} > x_2 + z_2\Big) \\
&\qquad \times \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&\sim \sum_{i=1}^{n}\sum_{j=1}^{n} \int\cdots\int_{\{0\le t_1\le\ldots\le t_n\le T, t_{n+1}>T\}} \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}\big(X_{1i}e^{-\delta t_i} > x_1 + z_1, X_{2j}e^{-\delta t_j} > x_2 + z_2\big) \\
&\qquad \times \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}). \qquad (3.23)
\end{aligned}$$

By $F_k \in \mathscr{S} \subset \mathscr{L}$, $\lim_{x\to\infty} \phi_k(x) = d_k,\ k = 1, 2$, and using the dominated convergence theorem, we have that for each $1 \leq i \leq n$,

$$\begin{aligned}
&\lim_{\min\{x_1,x_2\}\to\infty} \frac{\mathbb{P}(X_{1i} > x_1 + Z_1e^{\delta T}, X_{2i} > x_2 + Z_2e^{\delta T})}{\overline{F_1}(x_1)\overline{F_2}(x_2)} \\
&= \int_{0-}^{\infty}\int_{0-}^{\infty} \lim_{\min\{x_1,x_2\}\to\infty} \frac{\mathbb{P}(X_{1i} > x_1 + z_1e^{\delta T}, X_{2i} > x_2 + z_2e^{\delta T})}{\overline{F_1}(x_1)\overline{F_2}(x_2)} \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2) \\
&= 1 + rd_1d_2.
\end{aligned}$$

Thus, for each $1 \leq i \leq n$, any $\epsilon > 0$ and sufficiently large x_1, x_2,

$$\mathbb{P}(X_{1i} > x_1 + Z_1e^{\delta T}, X_{2i} > x_2 + Z_2e^{\delta T}) \geq (1 - \epsilon)(1 + rd_1d_2)\overline{F_1}(x_1)\overline{F_2}(x_2).$$

Using the above inequality we can get that for each $1 \leq i \leq n$,

$$\begin{aligned}
&\int\cdots\int_{\{0\leq t_1\leq\ldots\leq t_n\leq T, t_{n+1}>T\}} \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}\big(X_{1i}e^{-\delta t_i} > x_1 + z_1, X_{2i}e^{-\delta t_i} > x_2 + z_2\big) \\
&\qquad\qquad \times \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&\geq \int\cdots\int_{\{0\leq t_1\leq\ldots\leq t_n\leq T, t_{n+1}>T\}} \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}\big(X_{1i} > x_1e^{\delta t_i} + z_1e^{\delta T}, X_{2i} > x_2e^{\delta t_i} + z_2e^{\delta T}\big) \\
&\qquad\qquad \times \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&\geq (1-\epsilon)(1+rd_1d_2) \int\cdots\int_{\{0\leq t_1\leq\ldots\leq t_n\leq T, t_{n+1}>T\}} \mathbb{P}\big(X_{1i} > x_1e^{\delta t_i}\big)\mathbb{P}\big(X_{2i} > x_2e^{\delta t_i}\big) \\
&\qquad\qquad \times \mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&\geq (1-\epsilon)^2 \int\cdots\int_{\{0\leq t_1\leq\ldots\leq t_n\leq T, t_{n+1}>T\}} \mathbb{P}\big(X_{1i} > x_1e^{\delta t_i}, X_{2i} > x_2e^{\delta t_i}\big)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&= (1-\epsilon)^2\mathbb{P}\big(X_{1i}e^{-\delta\tau_i} > x_1, X_{2i}e^{-\delta\tau_i} > x_2, N(T) = n\big). \qquad (3.24)
\end{aligned}$$

In a similar but simpler way, we can obtain that for each $1 \leq i \neq j \leq n$, the above $\epsilon > 0$ and sufficiently large x_1, x_2,

$$\begin{aligned}
&\int\cdots\int_{\{0\leq t_1\leq\ldots\leq t_n\leq T, t_{n+1}>T\}} \int_{0-}^{\infty}\int_{0-}^{\infty} \mathbb{P}\big(X_{1i}e^{-\delta t_i} > x_1 + z_1, X_{2j}e^{-\delta t_j} > x_2 + z_2\big) \\
&\qquad\qquad \times \mathbb{P}(Z_1 \in \mathrm{d}z_1, Z_2 \in \mathrm{d}z_2)\mathbb{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{n+1} \in \mathrm{d}t_{n+1}) \\
&\geq (1-\epsilon)\mathbb{P}\big(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2, N(T) = n\big). \qquad (3.25)
\end{aligned}$$

Plugging (3.24) and (3.25) into (3.23) yields that

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1 + Z_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2 + Z_2, N(T) = n\Big) \\
&\quad \gtrsim (1-\epsilon)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}\big(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2, N(T) = n\big),
\end{aligned}$$

which means that the desired (3.22) holds by taking account of the arbitrariness of ϵ and the nonnegativity of Z_1 and Z_2. It ends the proof. □

Now we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. For any fixed $T \in \Lambda$, the finite-time ruin probability in (1.3) can be rewritten as

$$\begin{aligned}
\psi(\mathbf{x};T) = \mathbb{P}\Big(&\sup_{t\in[0,T]} \sum_{i=1}^{N(t)} \Big(X_{1i}e^{-\delta\tau_i} - \int_{0-}^{t} e^{-\delta s}C_1(\mathrm{d}s)\Big) > x_1, \\
&\sup_{t\in[0,T]} \sum_{j=1}^{N(t)} \Big(X_{2j}e^{-\delta\tau_j} - \int_{0-}^{t} e^{-\delta s}C_2(\mathrm{d}s)\Big) > x_2\Big). \qquad (3.26)
\end{aligned}$$

Hereafter, we denote by $\varphi(\mathbf{x};T)$ the right-hand side of relation (2.3). We firstly deal with the upper bound of (2.3). Clearly, for any fixed positive integer n_0,

$$\begin{aligned}\psi(\mathrm{x};T) &\le \mathbb{P}\Big(\sum_{i=1}^{N(T)} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{N(T)} X_{2j}e^{-\delta\tau_j} > x_2\Big)\\ &= \left(\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{\infty}\right)\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2, N(T)=n\Big)\\ &=: I_1 + I_2. \end{aligned} \tag{3.27}$$

As for I_2, note that $\{\theta_i,\ i\ge 2\}$ are i.i.d. nonnegative r.v's with common d.f. G, and independent of θ_1 with d.f. G_1. Hence, for any $\epsilon>0$, all $n\ge n_0+1$ and $x_1>0,\ x_2>0$,

$$\begin{aligned}&\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2, N(T)=n\Big)\\ &\le \mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_1} > x_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_1} > x_2, \tau_n\le T, \tau_{n+1}>T\Big)\\ &= \int_{0-}^{T}\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i} > x_1e^{\delta t}, \sum_{j=1}^{n} X_{2j} > x_2e^{\delta t}\Big)\mathbb{P}(N_0(T-t)=n-1)G_1(\mathrm{d}t)\\ &\le K_\epsilon(1+|r|\kappa_1\kappa_2)^n(1+\epsilon)^n\int_{0-}^{T}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})\mathbb{P}(N_0(T-t)=n-1)G_1(\mathrm{d}t) \end{aligned}\tag{3.28}$$

where we used Lemma 3.4 in the last step. By $\mathbb{E}\rho^{N_0(T)}<\infty$ for some $\rho>1+|r|\kappa_1\kappa_2$, for any $\varepsilon>0$, we can choose a sufficiently small $\epsilon>0$ and a sufficiently large integer n_0 such that

$$\begin{aligned}\max\Big\{&\mathbb{E}\Big((1+|r|\kappa_1\kappa_2)(1+\epsilon)\Big)^{N_0(T)}\mathbb{I}_{\{N_0(T)\ge n_0\}},\\ &\mathbb{E}\Big((N_0(T)+1)(N_0(T)+1+|r|b_1b_2)\Big)\mathbb{I}_{\{N_0(T)\ge n_0\}}\Big\}\le\varepsilon. \end{aligned}\tag{3.29}$$

Then, we obtain from (3.28) and (3.29) that

$$\begin{aligned}I_2 &\le K_\epsilon\int_{0-}^{T}\mathbb{E}\big((1+|r|\kappa_1\kappa_2)(1+\epsilon)\big)^{N_0(T-t)}\mathbb{I}_{\{N_0(T-t)\ge n_0\}}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})G_1(\mathrm{d}t)\\ &\le K_\epsilon\mathbb{E}\big((1+|r|\kappa_1\kappa_2)(1+\epsilon)\big)^{N_0(T)+1}\mathbb{I}_{\{N_0(T)\ge n_0\}}\int_{0-}^{T}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})G_1(\mathrm{d}t)\\ &\le K\varepsilon\varphi(\mathbf{x};T). \end{aligned}\tag{3.30}$$

Now we turn to I_1. In Lemma 3.5, take $Z_1=Z_2\equiv 0$, then, for the above fixed integer n_0 and any $\varepsilon>0$, we have that for sufficiently large x_1 and x_2,

$$\begin{aligned}I_1 &\le (1+\varepsilon)\sum_{n=1}^{n_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2j}e^{-\delta\tau_j}>x_2, N(T)=n)\\ &= (1+\varepsilon)\sum_{n=1}^{n_0}\left(\sum_{i=1}^{n}\sum_{j=i} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} + \sum_{i=2}^{n}\sum_{j=1}^{i-1}\right)\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2j}e^{-\delta\tau_j}>x_2, N(T)=n)\\ &=: (1+\varepsilon)(I_{11}+I_{12}+I_{13}). \end{aligned}\tag{3.31}$$

By the conditions of the theorem, we have

$$\begin{aligned}
I_{11} &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2i}e^{-\delta\tau_i}>x_2, N(T)=n)\\
&= \sum_{i=1}^{\infty}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2i}e^{-\delta\tau_i}>x_2, \tau_i\le T)\\
&= \int_{0-}^{T}\mathbb{P}(X_{1i}>x_1e^{\delta u}, X_{2i}>x_2e^{\delta u})\lambda(\mathrm{d}u)\\
&\sim (1+rd_1d_2)\int_{0-}^{T}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\lambda(\mathrm{d}u). \qquad (3.32)
\end{aligned}$$

As for I_{12}, since, for each $1\le i<j\le n\le n_0$, X_{1i} and X_{2j} are independent, and $\{N(t),\ t\ge 0\}$ is a delayed renewal process, we obtain

$$\begin{aligned}
I_{12} &\le \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2j}e^{-\delta(\tau_j-\tau_i)-\delta\tau_i}>x_2, \tau_j\le T)\\
&= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\int_{0-}^{T}\int_{0-}^{T-u}\mathbb{P}(X_{1i}>x_1e^{\delta u})\mathbb{P}(X_{2j}>x_2e^{\delta(u+v)})\mathbb{P}(\tau_j-\tau_i\in\mathrm{d}v)\mathbb{P}(\tau_i\in\mathrm{d}u)\\
&= \int_{0-}^{T}\int_{0-}^{T-u}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)})\lambda_0(\mathrm{d}v)\lambda(\mathrm{d}u). \qquad (3.33)
\end{aligned}$$

In the same manner, we can obtain

$$I_{13}\le \int_{0-}^{T}\int_{0-}^{T-u}\overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})\lambda_0(\mathrm{d}v)\lambda(\mathrm{d}u). \qquad (3.34)$$

Substituting (3.30)–(3.34) into (3.27), and by the arbitrariness of $\varepsilon>0$, leads to the upper bound of relation (2.3).

Finally, we investigate the lower bound of (2.3). By (3.26) and Lemma 3.5, for the above $\varepsilon>0$ and integer n_0 defined in (3.29), we derive that for sufficiently large x_1 and x_2,

$$\begin{aligned}
\psi(\mathrm{x};T) &\ge \sum_{n=1}^{n_0}\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1+\int_{0-}^{T}e^{-\delta s}C_1(\mathrm{d}s),\\
&\qquad \sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j}>x_2+\int_{0-}^{T}e^{-\delta s}C_2(\mathrm{d}s), N(T)=n\Big)\\
&\ge (1-\varepsilon)\left(\sum_{n=1}^{\infty}-\sum_{n=n_0+1}^{\infty}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2j}e^{-\delta\tau_j}>x_2, N(T)=n)\\
&=: (1-\varepsilon)(J_1-J_2). \qquad (3.35)
\end{aligned}$$

Similarly to the proofs of (3.32)–(3.34), we can obtain

$$J_1\sim\varphi(\mathbf{x};T). \qquad (3.36)$$

Similarly to (3.28) and (3.30), by Proposition 2.2 and (3.29), we obtain

$$
\begin{aligned}
J_2 &\leq \sum_{n=n_0+1}^{\infty}\sum_{i=1}^{n}\Bigg(\sum_{j=i}+\sum_{1\leq j\neq i\leq n}\Bigg)\mathbb{P}(X_{1i}e^{-\delta\tau_1}>x_1, X_{2j}e^{-\delta\tau_1}>x_2, N(T)=n)\\
&= \sum_{n=n_0+1}^{\infty}\sum_{i=1}^{n}\Bigg(\sum_{j=i}+\sum_{1\leq j\neq i\leq n}\Bigg)\int_{0-}^{T}\mathbb{P}(X_{1i}>x_1e^{\delta t}, X_{2j}>x_2e^{\delta t})\mathbb{P}(N_0(T-t)=n-1)G_1(\mathrm{d}t)\\
&\leq \sum_{n=n_0+1}^{\infty}\Big((1+|r|b_1b_2)n+n(n-1)\Big)\int_{0-}^{T}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})\mathbb{P}(N_0(T-t)=n-1)G_1(\mathrm{d}t)\\
&= \int_{0-}^{T}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})\mathbb{E}\Big((N_0(T-t)+1)(N_0(T-t)+1+|r|b_1b_2)\Big)\mathbb{1}_{\{N_0(T-t)\geq n_0\}}G_1(\mathrm{d}t)\\
&\leq \varepsilon\int_{0-}^{T}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})G_1(\mathrm{d}t)\\
&\leq \varepsilon\varphi(\mathbf{x};T). \qquad (3.37)
\end{aligned}
$$

Therefore, the desired lower bound of (2.3) follows from (3.35)–(3.37) and by the arbitrariness of $\varepsilon>0$. This completes the proof of the theorem. □

4 Proof of Theorem 2.2

Before the proof of Theorem 2.2, we prepare a series of lemmas. The first lemma is analogue to Lemma 3.5, but the claim arrival times τ_i's are not necessarily bounded on a finite interval.

Lemma 4.1. *Consider a two-dimensional risk model with a constant interest rate $\delta\geq 0$. Let $(X_1,X_2)^{\mathrm{T}}$ follow a bivariate Sarmanov distribution of the form (2.1) with marginal d.f.'s $F_1\in$ ERV$(-\alpha_1,-\beta_1)$ and $F_2\in$ ERV$(-\alpha_2,-\beta_2)$ for some $0<\alpha_1\leq\beta_1<\infty$ and $0<\alpha_2\leq\beta_2<\infty$. Assume that the limits $\lim_{x\to\infty}\phi_i(x)=d_i,\ i=1,2,$ exist. If $1+rd_1d_2>0$, then for any fixed $n\geq 1$,*

$$
\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j}>x_2\Big)\sim\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>x_1, X_{2j}e^{-\delta\tau_j}>x_2). \qquad (4.1)
$$

Proof. We firstly estimate the upper bound of (4.1). For any small $\Delta>0$,

$$
\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j}>x_2\Big)\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i}>(1-\Delta)x_1, X_{2j}e^{-\delta\tau_j}>(1-\Delta)x_2)\\
&\quad+\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j}>x_2,\\
&\qquad\bigcap_{i=1}^{n}\bigcap_{j=1}^{n}\Big(\{X_{1i}e^{-\delta\tau_i}\leq(1-\Delta)x_1\}\cup\{X_{2j}e^{-\delta\tau_j}\leq(1-\Delta)x_2\}\Big)\Big)\\
&=: I_1+I_2. \qquad (4.2)
\end{aligned}
$$

By the conditions of the lemma, we have that for each $1 \le i \le n$,

$$\begin{aligned}
&\mathbb{P}(X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1, X_{2i}e^{-\delta\tau_i} > (1-\Delta)x_2) \\
&\quad = \int_{0-}^{\infty} \mathbb{P}(X_{1i} > (1-\Delta)x_1 e^{\delta u}, X_{2i} > (1-\Delta)x_2 e^{\delta u})\mathbb{P}(\tau_i \in \mathrm{d}u) \\
&\quad \sim (1+rd_1d_2)\int_{0-}^{\infty} \overline{F_1}((1-\Delta)x_1e^{\delta u})\overline{F_2}((1-\Delta)x_2e^{\delta u})\mathbb{P}(\tau_i \in \mathrm{d}u) \\
&\quad \le (1+rd_1d_2)\sup_{u\ge 0}\frac{\overline{F_1}((1-\Delta)x_1e^{\delta u})\overline{F_2}((1-\Delta)x_2e^{\delta u})}{\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})}\cdot\int_{0-}^{\infty}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\mathbb{P}(\tau_i \in \mathrm{d}u) \\
&\quad \lesssim (1-\Delta)^{-(\beta_1+\beta_2)}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2i}e^{-\delta\tau_i} > x_2), \qquad (4.3)
\end{aligned}$$

and for each $1 \le i \ne j \le n$,

$$\begin{aligned}
&\mathbb{P}(X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1, X_{2j}e^{-\delta\tau_j} > (1-\Delta)x_2) \\
&\quad = \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{F_1}((1-\Delta)x_1e^{\delta u})\overline{F_2}((1-\Delta)x_2e^{\delta v})\mathbb{P}(\tau_i \in \mathrm{d}u, \tau_j \in \mathrm{d}v) \\
&\quad \le \sup_{u\ge 0}\frac{\overline{F_1}((1-\Delta)x_1e^{\delta u})}{\overline{F_1}(x_1e^{\delta u})}\sup_{v\ge 0}\frac{\overline{F_2}((1-\Delta)x_2e^{\delta v})}{\overline{F_2}(x_2e^{\delta v})}\cdot\int_{0-}^{\infty}\int_{0-}^{\infty}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta v})\mathbb{P}(\tau_i \in \mathrm{d}u, \tau_j \in \mathrm{d}v) \\
&\quad \lesssim (1-\Delta)^{-(\beta_1+\beta_2)}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2). \qquad (4.4)
\end{aligned}$$

Similarly to (4.3) and (4.4), we can also get that for each $1 \le i \le n$ and $1 \le j \le n$,

$$\mathbb{P}(X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1, X_{2j}e^{-\delta\tau_j} > (1-\Delta)x_2) \gtrsim (1-\Delta)^{-(\alpha_1+\alpha_2)}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2).$$

Thus, we derive that

$$\lim_{\Delta\downarrow 0}\lim_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{I_1}{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2)} = 1. \qquad (4.5)$$

As for I_2, we have

$$\begin{aligned}
I_2 \quad \le \quad & \sum_{k=1}^{n}\sum_{l=1}^{n}\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j} > x_2, X_{1k}e^{-\delta\tau_k} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_l} > \frac{x_2}{n}, \\
& \qquad \bigcap_{i=1}^{n}\bigcap_{j=1}^{n}\Big(\{X_{1i}e^{-\delta\tau_i} \le (1-\Delta)x_1\} \cup \{X_{2j}e^{-\delta\tau_j} \le (1-\Delta)x_2\}\Big)\Big) \\
\le \quad & \sum_{k=1}^{n}\sum_{l=1}^{n}\mathbb{P}\Big(\sum_{1\le i\ne k\le n}X_{1i}e^{-\delta\tau_i} > \Delta x_1, X_{1k}e^{-\delta\tau_k} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_l} > \frac{x_2}{n}\Big) \\
& + \sum_{k=1}^{n}\sum_{l=1}^{n}\mathbb{P}\Big(\sum_{1\le j\ne l\le n}X_{2j}e^{-\delta\tau_j} > \Delta x_2, X_{1k}e^{-\delta\tau_k} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_l} > \frac{x_2}{n}\Big) \\
\le \quad & \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{1\le i\ne k\le n}\mathbb{P}\Big(X_{1i} > \frac{\Delta x_1}{n-1}, X_{1k}e^{-\delta\tau_1} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_1} > \frac{x_2}{n}\Big) \\
& + \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{1\le j\ne l\le n}\mathbb{P}\Big(X_{2j} > \frac{\Delta x_2}{n-1}, X_{1k}e^{-\delta\tau_1} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_1} > \frac{x_2}{n}\Big) \\
=: \quad & I_{21} + I_{22}. \qquad (4.6)
\end{aligned}$$

We further divide I_{21} into three parts as

$$\begin{aligned}
I_{21} &= \Bigg(\sum_{k=1}^{n}\sum_{1\le l\ne k\le n}\sum_{1\le i\le n, i\ne k, i\ne l} + \sum_{k=1}^{n}\sum_{1\le l\ne k\le n}\sum_{i=l} + \sum_{k=1}^{n}\sum_{l=k}\sum_{1\le i\ne k\le n}\Bigg) \\
&\qquad \mathbb{P}\Big(X_{1i} > \frac{\Delta x_1}{n-1}, X_{1k}e^{-\delta\tau_1} > \frac{x_1}{n}, X_{2l}e^{-\delta\tau_1} > \frac{x_2}{n}\Big) \\
&=: I_{211} + I_{212} + I_{213}. \qquad (4.7)
\end{aligned}$$

From $F_i \in \mathrm{ERV}(-\alpha_i, -\beta_i) \subset \mathscr{D},\ i = 1, 2$, and $\mathbb{P}(X_{11} > x_1, X_{21} > x_2) \sim (1+rd_1d_2)\overline{F_1}(x_1)\overline{F_2}(x_2)$, it holds that

$$\begin{aligned}
I_{211} &= n(n-1)(n-2)\overline{F_1}\Big(\frac{\Delta x_1}{n-1}\Big)\int_{0-}^{\infty}\overline{F_1}\Big(\frac{x_1e^{\delta t}}{n}\Big)\overline{F_2}\Big(\frac{x_2e^{\delta t}}{n}\Big)\mathbb{P}(\tau_1 \in \mathrm{d}t) \\
&= o(1)\int_{0-}^{\infty}\overline{F_1}(x_1e^{\delta t})\overline{F_2}(x_2e^{\delta t})\mathbb{P}(\tau_1 \in \mathrm{d}t) \\
&= o(1)\mathbb{P}(X_{11}e^{-\delta\tau_1} > x_1, X_{21}e^{-\delta\tau_1} > x_2).
\end{aligned}$$

Similarly, we can also obtain $I_{21i} = o(1)\mathbb{P}(X_{11}e^{-\delta\tau_1} > x_1, X_{21}e^{-\delta\tau_1} > x_2),\ i = 2, 3$. Substituting these estimates into (4.7), we derive that

$$I_{21} = o(1)\mathbb{P}(X_{11}e^{-\delta\tau_1} > x_1, X_{21}e^{-\delta\tau_1} > x_2). \qquad (4.8)$$

We can use the same approach to get

$$I_{22} = o(1)\mathbb{P}(X_{11}e^{-\delta\tau_1} > x_1, X_{21}e^{-\delta\tau_1} > x_2). \qquad (4.9)$$

Combining (4.6), (4.8) and (4.9) leads to

$$I_2 = o(1)\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2). \qquad (4.10)$$

Therefore, the upper bound of (4.1) is derived from (4.2), (4.5) and (4.10).

We next deal with the lower bound of (4.1). Clearly, by Bonferroni's inequality,

$$
\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{n} X_{2j}e^{-\delta\tau_j} > x_2\Big) \\
&\geq \mathbb{P}\Big(\bigcup_{i=1}^{n}\{X_{1i}e^{-\delta\tau_i} > x_1\}, \bigcup_{j=1}^{n}\{X_{2j}e^{-\delta\tau_j} > x_2\}\Big) \\
&\geq \sum_{i=1}^{n}\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i} > x_1, \bigcup_{j=1}^{n}\{X_{2j}e^{-\delta\tau_j} > x_2\}\Big) \\
&\quad -\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\mathbb{P}\Big(X_{1k}e^{-\delta\tau_k} > x_1, X_{1l}e^{-\delta\tau_l} > x_1, \bigcup_{j=1}^{n}\{X_{2j}e^{-\delta\tau_j} > x_2\}\Big) \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2) \\
&\quad -\sum_{i=1}^{n}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2, X_{2l}e^{-\delta\tau_l} > x_2) \\
&\quad -\sum_{j=1}^{n}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\mathbb{P}(X_{1k}e^{-\delta\tau_k} > x_1, X_{1l}e^{-\delta\tau_l} > x_1, X_{2j}e^{-\delta\tau_j} > x_2) \\
&=: \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2) - J_1 - J_2. \qquad (4.11)
\end{aligned}
$$

We only estimate J_1, because J_2 can be done in the same manner. By the conditions of the lemma, we have

$$
\begin{aligned}
J_1 &\leq \sum_{i=1}^{n}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2, X_{2l} > x_2) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n-1}\sum_{k+1\leq l\neq i\leq n}\overline{F_2}(x_2)\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2) \\
&\quad +\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2, X_{2i} > x_2) \\
&\sim o(1)\sum_{i=1}^{n}\sum_{k=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2) \\
&\quad +(1+rd_1d_2)\overline{F_2}(x_2)\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2k}e^{-\delta\tau_k} > x_2) \\
&= o(1)\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2). \qquad (4.12)
\end{aligned}
$$

Therefore, the desired lower bound of (4.1) follows from (4.11) and (4.12). □

Lemma 4.2. *Under the conditions of Theorem 2.2, it holds that*

$$\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\mathbb{P}\left(\sum_{i=n+1}^{\infty}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=1}^{n}X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}$$
$$=\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\mathbb{P}\left(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=n+1}^{\infty}X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0, \tag{4.13}$$

and

$$\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\sum_{i=n+1}^{\infty}\sum_{j=1}^{n}\mathbb{P}\left(X_{1i}e^{-\delta\tau_i}>x_1,X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}$$
$$=\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}\mathbb{P}\left(X_{1i}e^{-\delta\tau_i}>x_1,X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0. \tag{4.14}$$

Proof. We only prove the second equality of (4.13), and relation (4.14) can be proved by using a similar but simpler method. For any $\epsilon>0$ and any integer n such that $\sum_{j=n+1}^{\infty}j^{-2}<1$, we derive that

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}e^{-\delta\tau_i}>x_1,\ \sum_{j=n+1}^{\infty}X_{2j}e^{-\delta\tau_j}>x_2\Big)\\
&\le\mathbb{P}\Big(\sum_{i=1}^{n}X_{1i}>x_1\Big)\mathbb{P}\Big(\bigcup_{j=n+1}^{\infty}\Big\{X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big\}\Big)\\
&\le K_\epsilon(1+\epsilon)^n\overline{F_1}(x_1)\sum_{j=n+1}^{\infty}\mathbb{P}\Big(X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big)\\
&\le K_\epsilon\overline{F_1}(x_1)\sum_{j=n+1}^{\infty}(1+\epsilon)^{j-1}\mathbb{P}\Big(X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big),
\end{aligned}\tag{4.15}$$

where we used the standard Kesten's inequality in the second step, because of $F_1\in\mathrm{ERV}(-\alpha_1,-\beta_1)\subset\mathscr{S}$. Choose two constants α_2' and β_2' satisfying $0<\alpha_2'<\alpha_2\le\beta_2<\beta_2'<\infty$. Then, by Proposition 2.1 (1), there exist two positive constants c_{F_2} and d_{F_2} such that Proposition 2.1 (1) holds for all $x\ge y\ge d_{F_2}$. For all $j\ge1$ and $x_2>0$, introduce three events $A_1(j,x_2)=\{j^{-2}e^{\delta\tau_j}\le d_{F_2}x_2^{-1}\}$, $A_2(j,x_2)=\{d_{F_2}x_2^{-1}<j^{-2}e^{\delta\tau_j}\le1\}$ and $A_3(j,x_2)=\{j^{-2}e^{\delta\tau_j}>1\}$. We divide the sum in the right-hand side of (4.15) into three parts as $I_1+I_2+I_3$ with

$$I_k=\sum_{j=n+1}^{\infty}(1+\epsilon)^{j-1}\mathbb{E}\Big(\mathbb{P}\Big(X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big|\tau_j\Big)\mathbb{I}_{A_k}\Big),\ k=1,2,3. \tag{4.16}$$

By applying Markov's inequality and choosing $\epsilon>0$ small enough such that $(1+\epsilon)\max\{\mathbb{E}(e^{-\delta\alpha_2'\theta_2}),\mathbb{E}(e^{-\delta\beta_2'\theta_2})\}<1$, we have

$$I_1\le\Big(\frac{x_2}{d_{F_2}}\Big)^{-\beta_2'}\sum_{j=1}^{\infty}j^{2\beta_2'}\mathbb{E}(e^{-\delta\beta_2'\theta_1})\cdot\Big((1+\epsilon)\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{j-1}=o(\overline{F_2}(x_2)), \tag{4.17}$$

where we used Proposition 2.1 (2) in the last step. By Proposition 2.1 (1), for all $x_2\ge d_{F_2}$, we obtain, respectively, that

$$I_2\le c_{F_2}\overline{F_2}(x_2)\sum_{j=n+1}^{\infty}j^{2\beta_2'}\mathbb{E}(e^{-\delta\beta_2'\theta_1})\cdot\Big((1+\epsilon)\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{j-1}, \tag{4.18}$$

and

$$I_3 \leq c_{F_2}\overline{F_2}(x_2)\sum_{j=n+1}^{\infty} j^{2\alpha_2'}\mathbb{E}(e^{-\delta\alpha_2'\theta_1})\cdot\Big((1+\epsilon)\mathbb{E}(e^{-\delta\alpha_2'\theta_2})\Big)^{j-1}, \tag{4.19}$$

which imply that

$$\lim_{n\to\infty}\limsup_{x_2\to\infty}\frac{I_2+I_3}{\overline{F_2}(x_2)}=0. \tag{4.20}$$

Plugging (4.16), (4.17) and (4.20) into (4.15) leads to the desired result. □

Lemma 4.3. *Under the conditions of Theorem 2.2, it holds that*

$$\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\mathbb{P}\left(\sum_{i=n+1}^{\infty}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=n+1}^{\infty}X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0, \tag{4.21}$$

$$\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\mathbb{P}\left(X_{1i}e^{-\delta\tau_i}>x_1,X_{2j}e^{-\delta\tau_j}>x_2\right)}{\overline{F_1}(x_1)\overline{F_2}(x_2)}=0. \tag{4.22}$$

Proof. We only give the proof of (4.21), and relation (4.22) can be derived in a similar way. Similarly to the proof of Lemma 4.2, for any integer n such that $\sum_{i=n+1}^{\infty}i^{-2}<1$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=n+1}^{\infty}X_{1i}e^{-\delta\tau_i}>x_1,\sum_{j=n+1}^{\infty}X_{2j}e^{-\delta\tau_j}>x_2\Big)\\
&\leq \mathbb{P}\Big(\bigcup_{i=n+1}^{\infty}\Big\{X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2}\Big\},\bigcup_{j=n+1}^{\infty}\Big\{X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big\}\Big)\\
&\leq \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2},X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big)\\
&= \sum_{i=n+1}^{\infty}\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2},X_{2i}e^{-\delta\tau_i}>\frac{x_2}{i^2}\Big)\\
&\quad+\left(\sum_{i=n+1}^{\infty}\sum_{j=i+1}^{\infty}+\sum_{j=n+1}^{\infty}\sum_{i=j+1}^{\infty}\right)\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2},X_{2j}e^{-\delta\tau_j}>\frac{x_2}{j^2}\Big)\\
&=: I_1+I_2+I_3.
\end{aligned} \tag{4.23}$$

As for I_1, by Proposition 2.2 and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
I_1 &\leq (1+|r|b_1b_2)\sum_{i=n+1}^{\infty}\int_{0-}^{\infty}\overline{F_1}\Big(\frac{x_1e^{\delta t}}{i^2}\Big)\overline{F_2}\Big(\frac{x_2e^{\delta t}}{i^2}\Big)\mathbb{P}(\tau_i\in \mathrm{d}t)\\
&\leq (1+|r|b_1b_2)\sum_{i=n+1}^{\infty}\left(\int_{0-}^{\infty}\Big(\overline{F_1}\Big(\frac{x_1e^{\delta t}}{i^2}\Big)\Big)^2\mathbb{P}(\tau_i\in \mathrm{d}t)\cdot\int_{0-}^{\infty}\Big(\overline{F_2}\Big(\frac{x_2e^{\delta t}}{i^2}\Big)\Big)^2\mathbb{P}(\tau_i\in \mathrm{d}t)\right)^{\frac{1}{2}}\\
&\leq (1+|r|b_1b_2)\left(\sum_{i=n+1}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_1}\Big(\frac{x_1e^{\delta t}}{i^2}\Big)\Big)^2\mathbb{P}(\tau_i\in \mathrm{d}t)\cdot\sum_{j=n+1}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_2}\Big(\frac{x_2e^{\delta t}}{j^2}\Big)\Big)^2\mathbb{P}(\tau_j\in \mathrm{d}t)\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.24}$$

Denote three events by $A_1(j,x_2) = \{j^{-2}e^{\delta\tau_j} \le d_{F_2}x_2^{-1}\}$, $A_2(j,x_2) = \{d_{F_2}x_2^{-1} < j^{-2}e^{\delta\tau_j} \le 1\}$ and $A_3(j,x_2) = \{j^{-2}e^{\delta\tau_j} > 1\}$ as in the proof of Lemma 4.2. Then,

$$\sum_{j=n+1}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_2}\Big(\frac{x_2e^{\delta t}}{j^2}\Big)\Big)^2\mathbb{P}(\tau_j\in \mathrm{d}t) = \sum_{j=n+1}^{\infty}\mathbb{E}\Big(\mathbb{P}^2\Big(X_{2j}e^{-\delta\tau_j} > \frac{x_2}{j^2}\Big|\tau_j\Big)(\mathbb{I}_{A_1}+\mathbb{I}_{A_2}+\mathbb{I}_{A_3})\Big) =: I_{11}+I_{12}+I_{13}. \tag{4.25}$$

Similarly to (4.17)–(4.19), for any α_2' and β_2' satisfying $0 < \alpha_2' < \alpha_2 \le \beta_2 < \beta_2' < \infty$, by Markov's inequality and Proposition 2.1 (1), (2), we have that for sufficiently large $x_2 \ge d_{F_2}$,

$$I_{11} \le \Big(\frac{x_2}{d_{F_2}}\Big)^{-2\beta_2'}\sum_{j=1}^{\infty}j^{4\beta_2'}\mathbb{E}(e^{-2\delta\beta_2'\theta_1})\cdot\Big(\mathbb{E}(e^{-2\delta\beta_2'\theta_2})\Big)^{j-1} = o((\overline{F_2}(x_2))^2), \tag{4.26}$$

$$I_{12} \le c_{F_2}^2(\overline{F_2}(x_2))^2\sum_{j=n+1}^{\infty}j^{4\beta_2'}\mathbb{E}(e^{-2\delta\beta_2'\theta_1})\cdot\Big(\mathbb{E}(e^{-2\delta\beta_2'\theta_2})\Big)^{j-1}, \tag{4.27}$$

$$I_{13} \le c_{F_2}^2(\overline{F_2}(x_2))^2\sum_{j=n+1}^{\infty}j^{4\alpha_2'}\mathbb{E}(e^{-2\delta\alpha_2'\theta_1})\cdot\Big(\mathbb{E}(e^{-2\delta\alpha_2'\theta_2})\Big)^{j-1}, \tag{4.28}$$

which, combined with (4.25), yield that

$$\lim_{n\to\infty}\limsup_{x_2\to\infty}\frac{\sum_{j=n+1}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_2}\Big(\frac{x_2e^{\delta t}}{j^2}\Big)\Big)^2\mathbb{P}(\tau_j\in \mathrm{d}t)}{(\overline{F_2}(x_2))^2} = 0. \tag{4.29}$$

In the same manner, we can prove that

$$\lim_{n\to\infty}\limsup_{x_1\to\infty}\frac{\sum_{j=n+1}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_1}\Big(\frac{x_1e^{\delta t}}{i^2}\Big)\Big)^2\mathbb{P}(\tau_i\in \mathrm{d}t)}{(\overline{F_1}(x_1))^2} = 0. \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.24), we conclude that

$$\lim_{n\to\infty}\limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}}\frac{I_1}{\overline{F_1}(x_1))\overline{F_2}(x_2))} = 0. \tag{4.31}$$

We mainly deal with I_2. For each $j > i \ge n+1$, since $\tau_j - \tau_i = \sum_{k=i+1}^{j}\theta_k$ is independent of τ_i, and has the d.f. $G^{(j-i)*}$, then

$$\begin{aligned} I_2 &= \sum_{i=n+1}^{\infty}\sum_{j=i+1}^{\infty}\int_{0-}^{\infty}\mathbb{P}\Big(X_{1i} > \frac{x_1e^{\delta t}}{i^2}\Big)\mathbb{P}\Big(X_{2j}e^{-\delta\sum_{k=i+1}^{j}\theta_k} > \frac{x_2e^{\delta t}}{j^2}\Big)\mathbb{P}(\tau_i\in \mathrm{d}t) \\ &\le \sum_{i=n+1}^{\infty}\left(\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i} > \frac{x_1}{i^2}\Big)\sum_{j=i+1}^{\infty}\mathbb{P}\Big(X_{2j}e^{-\delta\sum_{k=i+1}^{j}\theta_k} > \frac{x_2}{j^2}\Big)\right). \end{aligned} \tag{4.32}$$

For each $i \ge n+1$, write

$$\begin{aligned} &\sum_{j=i+1}^{\infty}\mathbb{P}\Big(X_{2j}e^{-\delta\sum_{k=i+1}^{j}\theta_k} > \frac{x_2}{j^2}\Big) \\ &= \sum_{j=i+1}^{\infty}\mathbb{E}\Big(\mathbb{P}\Big(X_{2j}e^{-\delta\sum_{k=i+1}^{j}\theta_k} > \frac{x_2}{j^2}\Big|\sum_{k=i+1}^{j}\theta_k\Big)(\mathbb{I}_{B_1}+\mathbb{I}_{B_2}+\mathbb{I}_{B_3})\Big) \\ &=: I_{21}+I_{22}+I_{23}, \end{aligned} \tag{4.33}$$

where the events $B_1(j,x_2)=\{j^{-2}e^{\delta\sum_{k=i+1}^{j}\theta_k}\le d_{F_2}x_2^{-1}\}$, $B_2(j,x_2)=\{d_{F_2}x_2^{-1}<j^{-2}e^{\delta\sum_{k=i+1}^{j}\theta_k}\le 1\}$ and $B_3(j,x_2)=\{j^{-2}e^{\delta\sum_{k=i+1}^{j}\theta_k}>1\}$. As done in (4.26), for the above $0<\alpha_2'<\alpha_2\le\beta_2<\beta_2'<\infty$, by Markov's inequality, C_r inequality and Proposition 2.1 (2), we have that for sufficiently large x_2,

$$\begin{aligned}
I_{21} &\le \Big(\frac{x_2}{d_{F_2}}\Big)^{-\beta_2'}\sum_{j=i+1}^{\infty} j^{2\beta_2'}\cdot\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{j-i}\\
&= \Big(\frac{x_2}{d_{F_2}}\Big)^{-\beta_2'}\sum_{k=1}^{\infty}(k+i)^{2\beta_2'}\cdot\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}\\
&\le K\Big(\frac{x_2}{d_{F_2}}\Big)^{-\beta_2'}\left(\sum_{k=1}^{\infty}k^{2\beta_2'}\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}+i^{2\beta_2'}\sum_{k=1}^{\infty}\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}\right)\\
&= o(\overline{F_2}(x_2))\cdot i^{2\beta_2'}. \qquad (4.34)
\end{aligned}$$

By Proposition 2.1 (1) and C_r inequality, we have that for all $x_2\ge d_{F_2}$,

$$\begin{aligned}
I_{22} &\le c_{F_2}\overline{F_2}(x_2)\sum_{k=1}^{\infty}(k+i)^{2\beta_2'}\cdot\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}\\
&\le K\overline{F_2}(x_2)\left(\sum_{k=1}^{\infty}k^{2\beta_2'}\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}+i^{2\beta_2'}\sum_{k=1}^{\infty}\Big(\mathbb{E}(e^{-\delta\beta_2'\theta_2})\Big)^{k}\right)\\
&\le Ki^{2\beta_2'}\overline{F_2}(x_2), \qquad (4.35)
\end{aligned}$$

and similarly, for $x_2\ge d_{F_2}$,

$$I_{23}\le Ki^{2\alpha_2'}\overline{F_2}(x_2). \qquad (4.36)$$

Plugging (4.34)–(4.36) into (4.33), we obtain that for sufficiently large $x_2\ge d_{F_2}$,

$$\sum_{j=i+1}^{\infty}\mathbb{P}\Big(X_{2j}e^{-\delta\sum_{k=i+1}^{j}\theta_k}>\frac{x_2}{j^2}\Big)\le Ki^{2\beta_2'}\overline{F_2}(x_2).$$

This, combined with (4.32), implies that for sufficiently large $x_2\ge d_{F_2}$,

$$\begin{aligned}
I_2 &\le K\overline{F_2}(x_2)\sum_{i=n+1}^{\infty}i^{2\beta_2'}\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2}\Big)\\
&= K\overline{F_2}(x_2)\sum_{i=n+1}^{\infty}i^{2\beta_2'}\mathbb{E}\left(\mathbb{P}\Big(X_{1i}e^{-\delta\tau_i}>\frac{x_1}{i^2}\Big|\tau_i\Big)(\mathbb{1}_{E_1}+\mathbb{1}_{E_2}+\mathbb{1}_{E_3})\right)\\
&=: K\overline{F_2}(x_2)(I_{21}'+I_{22}'+I_{23}'), \qquad (4.37)
\end{aligned}$$

where the events $E_1(i,x_1)=\{i^{-2}e^{\delta\tau_i}\le d_{F_1}x_1^{-1}\}$, $E_2(i,x_1)=\{d_{F_1}x_1^{-1}<i^{-2}e^{\delta\tau_i}\le 1\}$, $E_3(i,x_1)=\{i^{-2}e^{\delta\tau_i}>1\}$, and $c_{F_1},\ d_{F_1},\ 0<\alpha_1'<\alpha_1\le\beta_1<\beta_1'<\infty$ are some positive constants such that Proposition 2.1 (1) holds. Analogously to the proofs of (4.26)–(4.28), we derive from Markov's inequality and Proposition 2.1 (1), (2) that for sufficiently large $x_1 \geq d_{F_1}$,

$$I'_{21} \leq \Big(\frac{x_1}{d_{F_1}}\Big)^{-\beta'_1} \sum_{i=1}^{\infty} i^{2(\beta'_1+\beta'_2)} \mathbb{E}(e^{-\delta\beta'_1\theta_1}) \cdot \Big(\mathbb{E}(e^{-\delta\beta'_1\theta_2})\Big)^{i-1} = o(\overline{F_1}(x_1)),$$

$$I'_{22} \leq c_{F_1}\overline{F_1}(x_1) \sum_{i=n+1}^{\infty} i^{2(\beta'_1+\beta'_2)} \mathbb{E}(e^{-\delta\beta'_1\theta_1}) \cdot \Big(\mathbb{E}(e^{-\delta\beta'_1\theta_2})\Big)^{i-1},$$

$$I'_{23} \leq c_{F_1}\overline{F_1}(x_1) \sum_{i=n+1}^{\infty} i^{2(\alpha'_1+\alpha'_2)} \mathbb{E}(e^{-\delta\alpha'_1\theta_1}) \cdot \Big(\mathbb{E}(e^{-\delta\alpha'_1\theta_2})\Big)^{i-1},$$

which, together with (4.37), lead to

$$\lim_{n\to\infty} \limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}} \frac{I_2}{\overline{F_1}(x_1)\overline{F_2}(x_2)} = 0. \tag{4.38}$$

Along the same line, we can also prove that

$$\lim_{n\to\infty} \limsup_{\mathbf{x}\to(\infty,\infty)^{\mathrm{T}}} \frac{I_3}{\overline{F_1}(x_1)\overline{F_2}(x_2)} = 0. \tag{4.39}$$

Therefore, the desired relation (4.21) follows from (4.23), (4.31), (4.38) and (4.39). It ends the proof of the lemma. □

Now we merge Lemmas 4.1–4.3 into a united one, which plays an important role in the proof of Theorem 2.2.

Lemma 4.4. *Under the conditions of Theorem 2.2, it holds that*

$$\mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2\Big) \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2). \tag{4.40}$$

Proof. We firstly consider the upper bound of (4.40). For any $\Delta > 0$ and a positive integer n_0, which we shall specify later,

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2\Big)\\
&\leq \mathbb{P}\Big(\Big\{\sum_{i=1}^{n_0} X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1\Big\} \bigcup \Big\{\sum_{i=n_0+1}^{\infty} X_{1i}e^{-\delta\tau_i} > \Delta x_1\Big\},\\
&\qquad \Big\{\sum_{j=1}^{n_0} X_{2j}e^{-\delta\tau_j} > (1-\Delta)x_2\Big\} \bigcup \Big\{\sum_{j=n_0+1}^{\infty} X_{2j}e^{-\delta\tau_j} > \Delta x_2\Big\}\Big)\\
&\leq \mathbb{P}\Big(\sum_{i=1}^{n_0} X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1, \sum_{j=1}^{n_0} X_{2j}e^{-\delta\tau_j} > (1-\Delta)x_2\Big)\\
&\quad + \mathbb{P}\Big(\sum_{i=1}^{n_0} X_{1i}e^{-\delta\tau_i} > (1-\Delta)x_1, \sum_{j=n_0+1}^{\infty} X_{2j}e^{-\delta\tau_j} > \Delta x_2\Big)\\
&\quad + \mathbb{P}\Big(\sum_{i=n_0+1}^{\infty} X_{1i}e^{-\delta\tau_i} > \Delta x_1, \sum_{j=1}^{n_0} X_{2j}e^{-\delta\tau_j} > (1-\Delta)x_2\Big)\\
&\quad + \mathbb{P}\Big(\sum_{i=n_0+1}^{\infty} X_{1i}e^{-\delta\tau_i} > \Delta x_1, \sum_{j=n_0+1}^{\infty} X_{2j}e^{-\delta\tau_j} > \Delta x_2\Big)\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.41}$$

For any $\epsilon > 0$, by (4.13) in Lemma 4.2 and (4.21) in Lemma 4.3, there are a sufficiently large integer n_0 and a large constant x_0, such that for sufficiently large $x_i \geq x_0,\ i = 1, 2$,

$$\begin{aligned} I_2 + I_3 + I_4 &\leq \epsilon\Big(\overline{F_1}((1-\Delta)x_1)\overline{F_2}(\Delta x_2) + \overline{F_1}(\Delta x_1)\overline{F_2}((1-\Delta)x_2) + \overline{F_1}(\Delta x_1)\overline{F_2}(\Delta x_2)\Big) \\ &\lesssim \epsilon\Big((1-\Delta)^{-\beta_1}\Delta^{-\beta_2} + \Delta^{-\beta_1}(1-\Delta)^{-\beta_2} + \Delta^{-(\beta_1+\beta_2)}\Big)\overline{F_1}(x_1)\overline{F_2}(x_2). \end{aligned} \tag{4.42}$$

For arbitrarily fixed $t_0 > 0$ such that $p_0 = \mathbb{P}(\tau_1 \leq t_0) > 0$, by $F_i \in \mathrm{ERV}(-\alpha_i, -\beta_i),\ i = 1, 2$, it is easy to see that

$$\begin{aligned} \mathbb{P}(X_1 e^{\delta\tau_1} > x_1, X_2 e^{\delta\tau_1} > x_2) &\geq p_0 \mathbb{P}(X_1 e^{\delta t_0} > x_1, X_2 e^{\delta t_0} > x_2) \\ &\sim p_0(1 - r d_1 d_2)\overline{F_1}(x_1 e^{\delta t_0})\overline{F_2}(x_2 e^{\delta t_0}) \\ &\gtrsim p_0(1 - r d_1 d_2) e^{-\delta t_0(\beta_1+\beta_2)}\overline{F_1}(x_1)\overline{F_2}(x_2). \end{aligned} \tag{4.43}$$

This and (4.42) yield that

$$I_2 + I_3 + I_4 \lesssim K\epsilon \mathbb{P}(X_{11} e^{\delta\tau_1} > x_1, X_{21} e^{\delta\tau_1} > x_2). \tag{4.44}$$

As for I_1, by Lemma 4.1 and $F_i \in \mathrm{ERV}(-\alpha_i, -\beta_i),\ i = 1, 2$, we have

$$\begin{aligned} I_1 &\sim \sum_{i=1}^{n_0}\sum_{j=1}^{n_0} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > (1-\Delta)x_1, X_{2j} e^{-\delta\tau_j} > (1-\Delta)x_2) \\ &\sim (1 + r d_1 d_2)\sum_{i=1}^{n_0}\int_{0-}^{\infty} \overline{F_1}((1-\Delta)x_1 e^{\delta u})\overline{F_2}((1-\Delta)x_2 e^{\delta u})\mathbb{P}(\tau_i \in \mathrm{d}u) \\ &\quad + \sum_{i=1}^{n_0}\sum_{1\leq j\neq i\leq n_0}\int_{0-}^{\infty}\int_{0-}^{\infty} \overline{F_1}((1-\Delta)x_1 e^{\delta u})\overline{F_2}((1-\Delta)x_2 e^{\delta v})\mathbb{P}(\tau_i \in \mathrm{d}u, \tau_j \in \mathrm{d}v) \\ &\lesssim (1-\Delta)^{-(\beta_1+\beta_2)}\sum_{i=1}^{n_0}\sum_{j=1}^{n_0} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > x_1, X_{2j} e^{-\delta\tau_j} > x_2). \end{aligned} \tag{4.45}$$

Hence, we conclude from (4.41), (4.44) and (4.45) that

$$\begin{aligned} &\mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i} e^{-\delta\tau_i} > x_1, \sum_{j=1}^{\infty} X_{2j} e^{-\delta\tau_j} > x_2\Big) \\ &\quad \lesssim \Big((1-\Delta)^{-(\beta_1+\beta_2)} + K\epsilon\Big)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > x_1, X_{2j} e^{-\delta\tau_j} > x_2). \end{aligned} \tag{4.46}$$

Now we turn to the lower bound of (4.40). For the above $\epsilon > 0$, by (4.14) in Lemma 4.2, (4.22) in Lemma 4.3 and (4.43), there exist a sufficiently large integer n_0' and a large constant x_0', such that for all $x_i \geq x_0',\ i = 1, 2$,

$$\begin{aligned} \max\Bigg\{ &\sum_{i=1}^{n_0'}\sum_{j=n_0'+1}^{\infty} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > x_1, X_{2j} e^{-\delta\tau_j} > x_2), \sum_{i=n_0'+1}^{\infty}\sum_{j=1}^{n_0'} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > x_1, X_{2j} e^{-\delta\tau_j} > x_2), \\ &\sum_{i=n_0'+1}^{\infty}\sum_{j=n_0'+1}^{\infty} \mathbb{P}(X_{1i} e^{-\delta\tau_i} > x_1, X_{2j} e^{-\delta\tau_j} > x_2)\Bigg\} \leq \epsilon \mathbb{P}(X_{11} e^{\delta\tau_1} > x_1, X_{21} e^{\delta\tau_1} > x_2). \end{aligned} \tag{4.47}$$

Applying Lemma 4.1, we derive from (4.47) that

$$
\begin{aligned}
&\mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2\Big)\\
&\geq \mathbb{P}\Big(\sum_{i=1}^{n_0'} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{n_0'} X_{2j}e^{-\delta\tau_j} > x_2\Big)\\
&\sim \sum_{i=1}^{n_0'}\sum_{j=1}^{n_0'} \mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2)\\
&= \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{n_0'}\sum_{j=n_0'+1}^{\infty} - \sum_{i=n_0'+1}^{\infty}\sum_{j=1}^{n_0'} - \sum_{i=n_0'+1}^{\infty}\sum_{j=n_0'+1}^{\infty}\right)\mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2)\\
&\geq (1-3\epsilon)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2). \qquad (4.48)
\end{aligned}
$$

Therefore, the obtained upper and lower bounds (4.46) and (4.48) lead to the desired relation (4.40), by taking account of the arbitrariness of $\epsilon > 0$ and $\Delta > 0$. □

Lemma 4.5. *Under the conditions of Theorem 2.2, it holds that*

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{P}(X_{1i}e^{-\delta\tau_i} > x_1, X_{2j}e^{-\delta\tau_j} > x_2)\\
&\quad\sim \int_{0-}^{\infty}\int_{0-}^{\infty}\Big(\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta(u+v)}) + \overline{F_1}(x_1e^{\delta(u+v)})\overline{F_2}(x_2e^{\delta u})\Big)\lambda_0(\mathrm{d}v)\lambda(\mathrm{d}u)\\
&\qquad + (1+rd_1d_2)\int_{0-}^{\infty}\overline{F_1}(x_1e^{\delta u})\overline{F_2}(x_2e^{\delta u})\lambda(\mathrm{d}u). \qquad (4.49)
\end{aligned}
$$

Proof. The proof can be given along a similar line of (3.32)–(3.34). □

Finally, we prove Theorem 2.2.

Proof of Theorem 2.2. Denote the right-hand side of (4.49) by $\varphi(\mathbf{x};\infty)$. As in (3.26), by Lemmas 4.4 and 4.5, the infinite-time ruin probability can be bounded from above by

$$
\psi(\mathbf{x};\infty) \leq \mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1, \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2\Big) \sim \varphi(\mathbf{x};\infty).
$$

On the other hand, again by Lemmas 4.4 and 4.5,

$$
\begin{aligned}
\psi(\mathbf{x};\infty) &\geq \mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1 + \int_{0-}^{\infty} e^{-\delta t}C_1(\mathrm{d}t), \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2 + \int_{0-}^{\infty} e^{-\delta t}C_2(\mathrm{d}t)\Big)\\
&= \int_0^{\infty}\int_0^{\infty}\mathbb{P}\Big(\sum_{i=1}^{\infty} X_{1i}e^{-\delta\tau_i} > x_1 + u, \sum_{j=1}^{\infty} X_{2j}e^{-\delta\tau_j} > x_2 + v\Big)H(\mathrm{d}u,\mathrm{d}v)\\
&\sim \int_0^{\infty}\int_0^{\infty}\varphi((x_1+u, x_2+v)^{\mathrm{T}};\infty)H(\mathrm{d}u,\mathrm{d}v), \qquad (4.50)
\end{aligned}
$$

where $H(u,v)$ denotes the joint d.f. of $(\int_{0-}^{\infty} e^{-\delta t} C_1(\mathrm{d}t), \int_{0-}^{\infty} e^{-\delta t} C_2(\mathrm{d}t))^{\mathrm{T}}$. For any small $\Delta > 0$ and any fixed $u > 0,\ v > 0$, by $F_i \in \mathrm{ERV}(-\alpha_i, -\beta_i),\ i = 1, 2$, we have that

$$\begin{aligned} \varphi((x_1+u, x_2+v)^{\mathrm{T}}; \infty) &\geq \varphi((1+\Delta)\mathbf{x}; \infty) \\ &\gtrsim (1+\Delta)^{-(\beta_1+\beta_2)} \varphi(\mathbf{x}; \infty), \end{aligned}$$

from which and (4.50), Fatou's lemma gives that

$$\begin{aligned} \liminf_{\mathbf{x} \to (\infty,\infty)^{\mathrm{T}}} \frac{\psi(\mathbf{x}; \infty)}{\varphi(\mathbf{x}; \infty)} &\geq \int_0^\infty \int_0^\infty \liminf_{\mathbf{x} \to (\infty,\infty)^{\mathrm{T}}} \frac{\varphi((x_1+u, x_2+v)^{\mathrm{T}}; \infty)}{\varphi(\mathbf{x}; \infty)} H(\mathrm{d}u, \mathrm{d}v) \\ &\geq (1+\Delta)^{-(\beta_1+\beta_2)}. \end{aligned}$$

Hence, the desired lower bound is derived by the arbitrariness of $\Delta > 0$. This completes the proof of Theorem 2.2. □

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